A Simple Model of Mortgage Insurance

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This paper uses a two-period model to analyze the borrower's choice of an optimal time pattern of mortgage payments in a world where future house values are uncertain. Since a decline in values can make the borrower's equity negative, leading to default on the mortgage, lenders in the model will require the purchase of mortgage insurance. The premium on the insurance policy will depend on the riskiness of the mortgage, which in turn depends on the magnitude of the initial mortgage payment. Mortgages with large (small) first payments will carry low (high) insurance premiums. Taking this fact into account, the borrower decides on the optimal riskiness of his mortgage. Borrowers who discount the future heavily choose risky mortgages carrying high insurance premiums, while those who place a higher value on future consumption opt for less risky contracts carrying low (or zero) premiums.

INTRODUCTION

The private mortgage insurance industry, which serves to protect lenders from default losses on risky mortgages, has experienced rapid growth since 1970. Privately insured loans rose from 5.1% of total conventional lending in 1970 to 18.2% in 1980. In 1980, 70% of mortgages with loan-to-value ratios above 80% carried private insurance.1

Although growth of the private industry has made mortgage insurance more readily accessible to borrowers, the housing finance literature lacks an explicit treatment of the choice problem posed by the availability of insurance. The problem is a simple one: currently, the borrower can opt for a mortgage with a low loan-to-value ratio (no higher than 80%) and avoid the cost of insurance, or he can demand a riskier mortgage (a 90% loan, for example) and pay the corresponding insurance premiums. The necessity of insuring a risky mortgage makes it costly to enjoy the postponement of housing expenses afforded by such a loan. If the borrower, however, has a

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strong preference for present as opposed to future consumption, bearing the cost of insurance may be worthwhile.

The purpose of the present paper is to develop a simple theory of mortgage insurance that, among other things, illustrates this basic proposition. The theory is based on a two-period model of mortgage choice in which borrower default is explicitly treated.\(^2\) Since derivation of an explicit default rule allows the required insurance premium to be computed for any mortgage contract, the trade-off between mortgage riskiness and insurance cost emerges clearly in the model. Borrowers choose the optimal time path of payments taking this trade-off into account. At the outset, it must be recognized that the contribution of this study is conceptual rather than practical. Rather than offering an operational model capable of generating detailed policy recommendations, the paper presents an abstract framework designed to aid our understanding of the role of mortgage insurance.

Previously, Brueckner (1984) analyzed the optimal timing of mortgage payments in a world where future house prices are certain (deriving, in effect, the optimal graduated payment mortgage under certainty).\(^3\) The insurance model developed here extends Brueckner's discussion to the uncertainty case, where future house prices are random. While default risk was eliminated in the certainty model via the simple requirement of no negative amortization, with mortgage insurance playing no role, insurance is mandatory under uncertainty. The reason is that even when negative amortization is avoided, a precipitous decline in house prices can make the borrower's equity negative, exposing the lender to the risk of default. With borrowers required to purchase insurance, the mortgage choice problem assumes a different character than in the certainty case. Postponing mortgage payments by increasing the tilt of the payment stream is now costly to the borrower since a higher insurance premium is required to compensate for the greater riskiness of the mortgage (under certainty, the only constraint on tilt was the avoidance of negative amortization).

To begin the discussion, the next section of the paper introduces the basic two-period model and analyses the optimal mortgage under certainty. As well as generating results analogous to those of Brueckner (1984) for a discrete-time model, the certainty analysis provides a benchmark for the subsequent discussion. The third section adds uncertainty to the basic framework and analyzes the optimal mortgage problem with insurance. The fourth section offers conclusions.

Before proceeding with the analysis, it is important to note the relationship between the present work and earlier papers dealing with the economics of collateral. Barro (1976), Benjamin (1978), Harris (1978), and Smith (1980, 1982) recognize the importance of the initial loan-to-value ratio in determining the probability of default when future asset value is random.\(^4\) However, rather than compensating the lender for default risk by insurance, compensation occurs via the contract interest rate in their models.\(^5\) A risky
loan will require a high interest rate to be attractive to the lender. While this approach may help explain the general level of mortgage interest rates in the economy, it is not realistic when applied to individual mortgage contracts. For example, borrowers desiring relatively risky 90% loans are not quoted a higher interest rate in the existing market but rather are offered another avenue (i.e., insurance) for satisfying the lender's default concerns. Although the net result in both cases is to raise the level of house payments, the fact that insurance premiums are front-loaded in the payment stream makes the two approaches conceptually and analytically distinct. By analyzing the insurance approach to the problem of default risk, the present paper complements the other literature.

THE BASIC FRAMEWORK AND THE CERTAINTY SOLUTION

In the model, borrowers live for two periods, denoted zero and one. At the beginning of each period, income is received and housing costs are incurred. Income not spent on housing is saved until the end of the period, when purchases of a composite non-housing commodity are made. If the borrower still owns his house at the end of period one, the house is sold and the proceeds are used for non-housing consumption (there is no bequest motive in the model). The artifice of introducing a temporal distinction between the beginning and end of a period, which follows Schwab (1982), increases the versatility of a simple two-period framework.

In order to focus on issues of interest, housing consumption is fixed exogenously in the analysis. The purchase price of the fixed-size house is equal to $v$ in period zero and is assumed to remain at $v$ in period one under certainty (under uncertainty, the period-one price is random). In both cases, general inflation is absent in that the price of the non-housing commodity remains constant over time at a value of unity. The mortgage payments in the two periods are denoted $m_0$ and $m_1$, and these payments must satisfy

$$m_0 + m_1/(1+r) = v,$$  

where $r$ is the one-period interest rate. Since the time pattern of mortgage payments is chosen by the borrower in the model, no downpayment requirement is imposed (any downpayment is simply part of $m_0$). Letting $y_0$ and $y_1$ denote incomes in the two periods and recalling that income net of housing costs is saved until the end of each period, non-housing consumption equals

$$c_0 = (y_0 - m_0) (1+r)$$  

in period zero and

$$c_1 = (y_1 - m_1) (1+r) + v$$
in period one for a consumer who holds his mortgage to maturity (recall that the house is sold at the end of period one in this case). It should be noted in (2) and (3) that the option of interperiod (as opposed to intraperiod) saving has been denied to the consumer (under this option, \( c_0 \) would be reduced by a savings term \( w_0 \), with \( c_1 \) increased by \((1+r)w_0\)). The reason is that such saving has no effect on the optimal mortgage when the utility function has the form imposed below. Unsecured borrowing against future income, which would involve a negative \( w_0 \), is also ruled out (the reason, of course, is that the lender has no recourse in the event of default on such a loan).

A borrower who terminates his mortgage at the beginning of period one is assumed to enter the rental market, where the fixed-size house commands a rent of \( R \) per period. Under the assumption that houses last forever, \( R \) must satisfy \( v = \sum_{i=0}^{\infty} R(1+r)^{-i} \), or \( R = rv/(1+r) \). When the mortgage is terminated, the borrower follows one of two courses of action. If equity at the beginning of period one is positive, sale of the house and settlement of the mortgage is the mode of termination. When equity is negative, however, mortgage termination will occur via default (the house is abandoned without penalty to the borrower in this case). Equity at the beginning of period one is equal to the value of the house minus the balance due on the mortgage, or \( v - m_1 \).

To determine the circumstances under which mortgage termination will occur, it is necessary to compare period-one consumption levels with and without termination. Consider first the case where equity is nonnegative and termination involves sale of the house. Period-one consumption in this case is given by

\[
c_1 = (y_1 - rv/(1+r) + v - m_1)(1+r). \tag{4}
\]

Note that the second term in (4) represents the rental payment and that there are no proceeds from an end-of-period house sale, as in (3). Since rearrangement shows that (4) is in fact exactly equal to (3), it follows that when equity is nonnegative, mortgage termination has no effect on period-one consumption. While the borrower is therefore indifferent between terminating the mortgage and holding it to maturity, this is not true when equity is negative. In this situation, period-one consumption is given by

\[
c_1 = (y_1 - rv/(1+r))(1+r), \tag{5}
\]

which exceeds (3) when \( v - m_1 < 0 \). As a result, the borrower gains from termination, which in this case involves default on the mortgage.

The above discussion shows that the default rule generated by the present model is the familiar one: the borrower should default on his mortgage when equity is negative. Before considering the effect of this default rule on the structure of the mortgage contract, it is interesting to see how several
modifications of the model alter the rule. First, if the stigma of mortgage default causes the borrower to bear an (explicit or implicit) penalty equal to \( q \), then (repeating the above argument) default should occur when \( v - m_i < -q \). Equity has to be below zero by at least the amount of the penalty for default to be desirable. On the other hand, suppose \( q = 0 \) but that mortgage interest is tax deductible. Then, since the tax deduction is lost when default occurs, default will be desirable only when \( v - m_i < -t_i \), where \( t_i \) is the marginal tax rate and \( l_i \) is the interest component of \( m_i \). Since \( l_i = r m_i / (1 + r) \), the default rule in this case reduces to \( v < (1 - t_i (1 + r)) m_i \). Note that for default to be desirable, \( v \) must be a smaller fraction of \( m_i \) when \( t \) is high than when \( t \) is low. This suggests that under a progressive tax system, high-income borrowers (who have high \( t_i \)'s) will default less readily than low-income borrowers. Note that without such a tax effect, the income of the borrower plays no role in the default decision.

To evaluate the consequences of default in the basic model, the first step is to note that when default occurs, the lender is denied the payment \( m_i \) but recovers the house through foreclosure. Assuming that the foreclosure process imposes costs equal to \( s > 0 \), the lender’s loss from default is thus \( m_i + s - v > 0 \) (recall that \( v < m_i \) holds under default). In a world of certainty, the lender can easily avoid this loss by structuring the mortgage contract to remove any incentive for default. In particular, the possibility of negative equity must be ruled out by requiring that the mortgage payment stream satisfy \( m_i \leq v \), or equivalently \( m_0 \geq rv / (1 + r) \) using (1). The latter inequality states that the first mortgage payment must at least cover interest on the original loan balance, ruling out negative amortization (this requirement was referred to as the “default risk constraint” in Brueckner (1984)). When the default risk constraint is satisfied, default will never occur in a world of certainty.

The last step in the treatment of the certainty case is analysis of the optimal mortgage contract from the borrower’s point of view. Although this is easily done using a general utility function, the analysis of the uncertainty case in the third section of this paper is inconclusive unless the utility function has the risk-neutral form \( u(c_0, c_1) = c_0 + c_1 / (1 + \theta) \), where \( \theta > 0 \) is the subjective discount rate (note that since \( v \) is fixed, it can be suppressed). Substituting (2) and (3) into the above function and eliminating \( m_0 \) using (1), utility equals

\[
(1 + r)(y_0 + y_1 / (1 + \theta)) - \frac{\theta + r}{1 + \theta} \frac{(1 + \theta)}{1 + \theta} v + \frac{\theta - r}{1 + \theta} m_1
\]

The borrower’s problem is to maximize this expression by choice of \( m_i \) subject to the constraint \( m_i \leq v \). The solution is immediate: when \( \theta > r \), the optimal mortgage has \( m_i = v \) and \( m_0 = rv / (1 + r) \) (the mortgage is a 100% loan with interest only paid in period zero); when \( \theta < r \), a mortgage with \( m_i = 0 \) and \( m_0 = v \) is optimal (the house is actually purchased outright); when \( \theta = r \), all feasible mortgages are equally attractive. The intuition behind these results is straight-
forward. When $\theta$ is high, the future is lowly valued and it is optimal to skew consumption toward period zero via the highest possible $m_1$. When $\theta$ is low, future consumption is highly valued and it is optimal to pay for the house entirely in period zero, allowing the largest possible non-housing consumption in period one.

With the certainty solution as a benchmark, the analysis now turns to the uncertainty case, where default risk is pervasive and mortgage insurance plays a key role.

**THE MORTGAGE CONTRACT UNDER UNCERTAINTY**

Under uncertainty, the period-one house value is a random variable $\tilde{v}$ rather than a fixed quantity. The minimum and maximum house values are denoted $a$ and $b$ respectively, and $\tilde{v}$'s probability density function, which is positive on $(a,b)$, is denoted $f$. The density $f$ is assumed for simplicity to be single-peaked, symmetric, and differentiable, with mean (and mode) equal to $v$ (this implies zero expected appreciation in house prices).

As shown in the previous section, mortgage default will occur when the period-one house value is less than $m_1$. The probability of default is therefore the probability that $\tilde{v}$ is less than $m_1$, or $\int_a^{m_1} f(\tilde{v})d\tilde{v}$. Note that this integral equals zero for $m_1 < a$ (indicating that default will not occur if $m_1$ lies below the range of possible house values) but is positive otherwise. When default occurs, the lender's loss in present value terms is given by $(m_1 + s - \tilde{v})/(1+r)$, and the expected present value of the loss is equal to

$$x = \frac{1}{1+r} \int_a^{m_1} (m_1 + s - \tilde{v}) f(\tilde{v}) d\tilde{v}$$

(7) equals zero for $m_1 \leq a$.

In a competitive mortgage market, the prospect of earning an expected present value of profit equal to zero will induce the (risk-neutral) lender to offer any given mortgage. This means that to secure a mortgage with a particular value of $m_1$, the borrower must make an up-front payment to the lender equal to his expected loss, given by $x$ in (7). Equivalently, the borrower can purchase a mortgage insurance policy that guarantees full compensation to the lender in the event of default. The premium on such a policy is given by (7). Note that since losses are fully covered under all possible mortgages, the default risk constraint of the previous section plays no role. A borrower is free to choose an $m_1$ greater than $v$ provided he pays the appropriate insurance premium. The effect of a higher $m_1$ on the magnitude of the premium is found by differentiating (7). Using Leibniz' rule, this yields
Since (8) is positive for \( m_1 > a \), the insurance premium increases with \( m_1 \) in this range. The reason, of course, is that a higher \( m_1 \) makes the mortgage riskier, with higher expected losses.

In deciding on the best time pattern of mortgage payments, the borrower recognizes that the required insurance premium rises with \( m_1 \). In addition, he realizes that period-one consumption will depend on whether or not he defaults on the mortgage. If \( \tilde{V} \geq m_1 \), so that default is not desirable, consumption \( c_1 \) is given by (3) with \( v \) (the proceeds from the end-of-period house sale) replaced by \( \tilde{V} \). If \( \tilde{V} < m_1 \), then default is optimal and \( c_1 \) is given by (5) with \( v \) replaced by \( \tilde{V} \) (the rent payment becomes random and equal to \( r\tilde{V}/(1+r)^2 \)). Substituting the appropriate consumption expressions into the linear utility function of the previous section and computing expected values, expected discount utility for the borrower is equal to

\[
\begin{align*}
&\left( y_0 - m_0 - x \right) (1+r) + \frac{1}{1+\theta} \int \left[ \frac{(y_1 - m_1)(1+r) + \tilde{V}}{m_1} \right] f(d\tilde{V}) \\
&+ \frac{1}{1+\theta} \int \left[ \frac{(y_1 - r\tilde{V}/(1+r)}{1+r} f(d\tilde{V}).
\end{align*}
\]

To express utility solely in terms of \( m_1 \), (7) is used to substitute for the insurance premium \( x \) in (9) and \( m_0 \) is eliminated using (1). After rearrangement, (9) then reduces to

\[
\begin{align*}
&\left( y_0 - m_0 - x \right) (1+r) + \frac{1}{1+\theta} \int \left[ \frac{(y_1 - m_1)(1+r) + \tilde{V}}{m_1} \right] f(d\tilde{V}) \\
&+ \frac{1}{1+\theta} \int \left[ \frac{(y_1 - r\tilde{V}/(1+r)}{1+r} f(d\tilde{V}),
\end{align*}
\]

where \( \Omega \) represents the first two terms in (6). Note that the term in brackets in (10) is the expected period-one return to the lender, while the last term is minus the expected foreclosure cost.

The borrower's goal is to choose \( m_1 \) to maximize expected utility as given by (10). Note that the optimal \( m_1 \) can range from zero all the way up to \( (1+r)v \) (the latter value results from setting \( m_0 = 0 \)). Using Leibniz' rule, the derivative of (10) with respect to \( m_1 \) equals

\[
\frac{\theta-r}{1+\theta} \int \frac{\tilde{V}f(d\tilde{V}) - sf(m_1)}{m_1}.
\]
When $0 < r$, (11) is negative, indicating that expected utility is a decreasing function of $m_i$. Since the smallest possible $m_i$ is best in this case, $m_i = 0$ is optimal. The outcome in the $0 < r$ case is therefore the same as in the certainty model, with the house purchased outright. The reason for the identical results is straightforward. In the certainty case, a low $\theta$ (a high valuation for future consumption) impelled the borrower to minimize $m_i$ in pursuit of the highest possible $c_i$. Since a low $m_i$ carries the additional benefit of a low (ultimately zero) insurance premium under uncertainty, the borrower has an added incentive to make $m_i$ as small as possible.\(^{13}\)

While the low-$\theta$ borrower thus exposes the lender to no risk (and pays no insurance premium), the outcome will be different for a borrower whose $\theta$ exceeds $r$. In this case, the first term in (11) will be positive, making the entire expression positive for $m_i \leq \alpha$. This means that expected utility increases with $m_i$ up to $\alpha$, implying that the optimal $m_i$ exceeds $\alpha$.\(^{14}\) How far above $\alpha$ the optimum lies depends on the magnitudes of $\theta$, $r$, and $s$ and on the nature of the density $f$. If these factors combine to make (11) positive for all $m_i$ between $\alpha$ and $(1+r)v$, then expected utility is monotonically increasing in $m_i$ and the riskiest possible mortgage, where $m_i$ assumes the maximal value $(1+r)v$, is optimal. This is the analog in the present model to the corner solution $m_i = v$ in the certainty case. This outcome indicates that in spite of the presence of a penalty for postponing mortgage payments (the insurance premium), the borrower nevertheless finds it optimal to postpone to the maximum extent possible, setting $m_0 = 0$ and paying a hefty premium $x$ to achieve the highest possible $m_i$. It can be shown that when (11) is everywhere positive, setting $m_i$ equal to $(1+r)v$ in fact minimizes $m_0 + x$ and thus leads to the highest possible period-zero consumption.\(^{15}\)

A more interesting outcome is where the presence of the insurance premium tempers the borrower's desire to postpone his mortgage payments, leading to an optimal mortgage of intermediate riskiness. For this outcome to obtain, the borrower's optimization problem must have an interior solution, with (11) equal to zero at some $m_i$ less than $(1+r)v$. Provided the relevant second-order condition holds,\(^{16}\) such a solution (if one exists) will be unique and will represent the global optimum.

It is clearly important to know the circumstances under which an interior optimum will emerge in the model. While the necessary and sufficient condition for this outcome is that (11) evaluated at $m_i = (1+r)v$ be negative, it is possible to derive a more intuitively appealing condition that guarantees the existence of an interior optimum satisfying $m_i < v$. This condition comes from noting that since $v$ is the mode of $f$ and $f$ is symmetric, the relationships

$$v \frac{b}{a} \frac{v}{V}$$

$$vf(v) \geq \int_a v f(x) dx = \int_a v f(x) dx$$

must hold (this can be seen from a graph of the
density function. Given \( \nu (v) > \int f(v) dv \), it follows that if the inequality
\[
s \geq (\theta - r) \frac{v}{(1 + \theta)},
\]
is satisfied, then \( s f(v) > \frac{\theta - r}{1 + \theta} \int f(v) dv \) also holds, implying that (11) evaluated at \( m_i = v \) is negative. This in turn implies that (11) equals zero (and an interior optimum exists) at a value of \( m_i \) below \( v \). Thus, when the foreclosure cost \( s \) exceeds a fraction \( (\theta - r)/(1 + \theta) \) of the period-zero house value, the mortgage insurance premium (which in part reflects the size of \( s \)) will be punitive enough to temper the borrower's desire to postpone his housing costs, leading to an optimal mortgage of intermediate riskiness. Note that the resulting mortgage payment stream shows less upward tilt than in the certainty case (\( m_0 > \frac{rv}{1+r} \) and \( m_i < v \) hold whereas \( m_0 = \frac{rv}{1+r} \) and \( m_i = v \) held before).

The above discussion has exposed the fundamental differences between the mortgage choice problems with and without uncertain future house values. Under certainty, the borrower could costlessly increase the upward tilt of his mortgage payment stream as long as the default risk constraint was not violated. Under uncertainty, however, any increase in the stream's tilt raises the default probability and hence the required insurance premium. In appraising the desirability of a greater tilt, the borrower must take the higher premium into account while recognizing that his chances of actually making the higher future mortgage payment diminish due to the greater likelihood that default will be optimal. While these joint considerations will not necessarily reduce the tilt of the payment stream relative to the certainty case, such an outcome is guaranteed when foreclosure costs are sufficiently high.

The last step in the analysis is to ascertain how the features of the optimal mortgage depend on the various parameters of the model. It is natural to consider first the impact of borrower characteristics. The only relevant borrower characteristic in the present model is, of course, the discount rate \( \theta \), and the effect on \( m_i \) of an increase in \( \theta \) is found by totally differentiating (11). This yields
\[
\frac{\partial m_i}{\partial \theta} = - \frac{(1+r)}{(1+\theta)} \frac{\Lambda}{m_i} \int f(v) dv > 0,
\]
where \( \Lambda \) is the reciprocal of the negative second-derivative of (10) (see Note 16). Equation (14) shows that when the borrower places a lower value on period-one consumption (when \( \theta \) rises), a greater transfer of purchasing power toward period zero is desirable, raising the optimal \( m_i \). Note that
since the insurance premium paid by the borrower rises, the higher $\theta$
effectively increases the demand for mortgage insurance.

Other parametric changes alter the optimal mortgage by affecting the supply side of the market. An increase in the foreclosure cost $s$, for example, raises the required insurance premium for any $m_i$ without affecting borrower preferences. Totally differentiating (11) yields

$$\frac{\partial m_i}{\partial s} = \Lambda f(m_i) < 0,$$

indicating that the optimal mortgage becomes less risky as the higher foreclosure cost drives up the cost of insurance. Since $m_i$ falls as $s$ rises, the actual insurance premium paid at the optimum can either rise or fall (see (7)).

An increase in $r$ generates both supply and demand effects. On the supply side, a higher $r$ reduces the insurance premium in (7) by lowering the present value of default losses. A higher $r$ also directly affects the consumer by raising the return to intraperiod saving as well as raising period-one rents. The net effect is to decrease the optimal $m_i$, with

$$\frac{\partial m_i}{\partial r} = \frac{\Lambda}{1+\theta} \int_{m_i}^b f(d\tilde{V}) < 0$$

by (11). Note that since $m_i$ falls as $r$ rises, the insurance premium paid at the optimum falls.

A final change that affects both borrower and lender is a shift in the distribution of period-one house values. Rewriting $f(v)$ as $f(v - \alpha)$, where $\alpha$ is a shift parameter, a shape-preserving rightward shift of the density function corresponds to an increase in $\alpha$. Such a shift raises the expected period-one house value and reduces the default probability for any value of $m_i$. Noting that the upper limit of integration in (11) must be replaced by $b + \alpha$, differentiation yields

$$\frac{\partial m_i}{\partial \alpha} = 1,$$

indicating that a rightward shift of the density raises $m_i$ by exactly the amount of the shift. This outcome makes sense given that with higher period-one house values, any given mortgage becomes less risky, reducing $x$ in (7) and enticing the consumer to increase $m_i$.

As a final exercise, it is interesting to solve for the optimal mortgage assuming a particular form for the density $f$. When $f$ is uniform on the interval $[v-\epsilon, v+\epsilon]$, so that $f = 1/2\epsilon$ in this range, \(11\) (11) yields

$$m_i = v + \epsilon - \frac{1+\theta}{\theta-r} s,$$
indicating that $m_i$ equals the maximum house value minus a multiple of $s$. While the above comparative static results can be verified by direct calculation, (18) yields the additional implication that $m_i$ rises as the variance of $v$ increases (as $\varepsilon$ rises). This relationship does not necessarily hold, however, when $f$ is nonuniform.$^{22,23}$

CONCLUSION

This paper has analyzed the borrower's choice of an optimal mortgage contract in the face of a market trade-off between mortgage riskiness and the cost of insurance. The analysis showed how the borrower's desire for mortgage insurance derives from the more fundamental demand for mortgage riskiness, with time preferences playing a key role. It was shown that borrowers who place a low value on future consumption demand risky mortgages and hence large insurance policies, while borrowers who value the future more highly opt for less risky mortgages carrying low (or even zero) insurance premiums. The analysis also highlighted the importance of the level of foreclosure costs and the distribution of future house prices in determining the riskiness and associated insurance cost of the optimal mortgage.

An important goal for future research would be to construct a model of mortgage insurance based on a more realistic portrayal of the consumer's incentive to default on a mortgage contract. While it was shown that the assumption of costless default can be modified by introducing a subjective default penalty, this is an inadequate way of capturing the powerful incentives against default that exist in the real world. Since these incentives mainly hinge on impairment of future borrowing power, a formal treatment would require extension of the borrower's time horizon beyond a simple two-period framework. Construction of such a model would no doubt provide important additional insights.

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NOTES

1. These data are taken from Swan (1982).
4. Since these models involve a single loan repayment, the time profile of payments is not an issue.
6. An equivalent assumption is that the consumer purchases outright a new house that is sold at the end of the period (see Note 7).

7. When a new house is purchased outright instead of rented (see Note 6), period-one consumption is \((y_1 - v + (v-m_i))(1+r) + v\), an expression that reduces to (4). A similar comparison holds under default.

8. The interest payment is that amount which leaves the balance due on the mortgage unchanged. In other words, if the balance due starts at \(B\), then \(l\) satisfies \(l + B/(1+r) = B\). When \(B = m_i\), \(l\) equals \(rm_i/(1+r)\). When \(B = v\), then \(l = rv/(1+r)\). The latter expression gives the interest component of \(m_0\).

9. It can be shown that the utility level under the optimal mortgage is at least as high as that afforded by continuous renting, a relationship that also holds under uncertainty.

10. To avoid inessential complications, \(b\) is assumed to exceed the largest possible value of \(m_1\), which equals \((1+r)v\).

11. In actual mortgage markets, insurance is not required when the loan-to-value ratio is 80% or less. It is interesting to note that in order for this outcome to emerge in the present stylized model, the \(m_1\) associated with an 80% conventional loan must be exactly equal to \(a\), the minimum house value (referring to (7), \(m_1 = a\) gives \(x = 0\). Using this fact, it is possible to calculate what the minimum house value must be in order for lenders to offer 80% loans without insurance. Since the constant principal and interest payment \(p\) under a conventional 80% mortgage must satisfy \(p + p/(1+r) = .8v\), it follows that \(m_1 = p = .8(1+r)v/(2+r)\) (note that \(m_0 = .2v + p\)). The requirement \(m_1 = a\) therefore implies that the period-one house value is expected to be no smaller than 40% of \(v\) (note \((1+r)/(2+r) = .5\). While the lack of realism of the model means that this conclusion cannot be taken seriously, the calculation is nevertheless instructive.

12. The implicit assumption that house values are equal to the capitalized value of current rents requires myopic expectations on the part of investors.

13. When \(\theta = r\), the optimal \(m_1\) lies between zero and \(a\) (all \(m_1\)'s in this range yield the same expected utility).

14. Recall that \(f = 0\) for \(m_1 \leq a\). Note that if \(f\) is not continuous at \(a\) (so that \(f(a) > 0\) holds), then while (11) would be zero for \(m_1 < a\), the expression could be negative for \(m_1 \geq a\), implying that \(m_1 = a\) is optimal.

15. Using (1) and (7), \(m_0 + x\) equals

\[
\int_a^{m_1} \left( v - m_1/(1+r) + \int_a^{m_1} (m_1 + s - \bar{v})f_d\bar{v}/(1+r) \right). \tag{F1}
\]

Using (F1), the derivative of \(m_0 + x\) with respect to \(m_1\) equals

\[
-FI = \frac{b}{m_1} - \left[ \int f_d\bar{v} - s f(m_1) \right]. \tag{F2}
\]

Since \((\theta-r)/(1+\theta) < 1\), it follows that the expression in brackets in (F2) exceeds (11). This in turn implies that when (11) is positive for all \(m_1 \leq (1+r)v\), (F2) is negative in this range, indicating that \(m_1 = (1+r)v\) minimizes \(m_0 + x\).

It should also be noted that (F2) must be negative at any solution to the optimization problem. Unless negativity holds, housing costs in both periods \((m_0 + x\) and \(m_1\)) could be reduced by lowering \(m_1\). While it has been shown that this requirement is satisfied at the above corner solution, it is easy to see that (F2) is also negative at the \(m_1 = 0\) solution in the \(\theta < r\) case and at the interior solution considered below.

16. The second-order condition requires that the objective function is strictly concave on the interval \([a, (1+r)v]\) (the function is linear on \([0, a]\)). Differentiating (11), this requires

\[
-\left( \frac{\theta-r}{\theta} \right) f(m_1) + sf(m_1) < 0. \tag{F3}
\]
Since $v$ is the mode of $f$, $f'(m_i) \geq 0$ holds for $a < m_i \leq v$ and (F3) is satisfied on this interval. Since $f' < 0$ holds above $v$, however, (F3) need not be satisfied on the interval $[v, (1+r)v]$. The inequality is nevertheless assumed to hold in this range.

17. The inequality follows because $vf(v)$ equals the area of a box with its southwest corner at the origin and its northeast corner at the peak of $f$ (this area exceeds the area under $f$ up to $v$). The equality follows since $f$ is symmetric around $v$.

18. This statement applies, of course, when $m_i \geq a$.

19. Differentiating the modified equation (11) with respect to $a$ yields

$$
\frac{b+a}{1+r} \left[ f(b) - \int_{m_i}^{b+\alpha} f(\tilde{v} - \alpha) d\tilde{v} \right] + sf'(m_i - \alpha) =
$$

$$
\frac{b+a}{1+r} \left[ f(b) - f(\tilde{v} - \alpha) \right] + sf'(m_i - \alpha).
$$

(E2)

20. Note that the uniform distribution has no peak and does not satisfy the smoothness requirements used in the analysis (it is discontinuous at $v-\varepsilon$ and $v+\varepsilon$).

21. Note that when (13) holds, (18) is less than or equal to $\varepsilon$. Since $v > \varepsilon$ must hold for the lower support of $f$ to be positive, the result that the interior solution is less than $v$ when (13) holds is verified.

22. A diagrammatic argument shows that when $f$ is single-peaked, an increase in the variance of the distribution has an ambiguous effect on the value of (11).

23. In the model of the second section of this paper, easy calculations show that housing investors earn zero profit, regardless of whether they are self-financing or use mortgage finance. While self-financing also yields zero expected profit in the uncertainty case as long as the expected value of $\tilde{v}$ equals $v$, it is important to verify that mortgage-based housing investment similarly earns zero expected profit under this assumption. The first step is to note that the default rule for a housing investor is identical to the one derived for consumers (period-one default occurs when equity is negative). This follows because period-one profit is $(rv/(1+r) - m_i - (1+r) + \tilde{v}$ when the mortgage is held to maturity and $(\max[\tilde{v} - m_i, 0]) (1+r)$ otherwise. Using this result and noting that period-zero profit is $(rv/(1+r) - m_0 - x) (1+r)$, the expected present value of profit equals $rv + Ev - (1+r)m_0 - m_i$, where $E$ denotes expected value. Under the maintained assumption $Ev = v$, this expression equals zero.
REFERENCES


